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Topological quantum field theory in quaternionic geometry

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Abstract

A topological quantum field theory on a $4k$-dimensional manifold M admitting an almost quaternionic structure is proposed. Expectation values of certain operators on M are proved to be independent of the choice of an almost quaternionic structure used in calculations and thus carry only smooth information about M. These invariants are explicitly expressed as integrals of differential forms over the instanton moduli space associated with a chosen almost quaternionic structure. When M admits a hyperKähler structure the topological field theory has three additional supersymmetries which induce three complex structures on the associated instanton moduli space proving thus that the latter is a hypercomplex manifold. In this case an analogue of the four-dimensional Donaldson map is constructed which provides a number of candidates for new invariants of $4k$-dimensional hypercomplex structures. In the case $k = 1$ the proposed topological theory on an almost quaternionic manifold reproduces the Witten interpretation of the four-dimensional Donaldson invariants.

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1. Introduction

Recently Witten [25] proposed a quantum field theory interpretation of the famous Donaldson invariants for smooth four-dimensional manifolds which

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appear to be extremely powerful in distinguishing different differentiable structures. The main topic of this paper is to construct a generalization of Witten's topological field theory to $4k$ dimensions, $k \geq 1$, and to use quantum field theory techniques for constructing Donaldson-like (smooth) invariants of a $4k$-dimensional manifold M admitting an almost quaternionic structure.

In our approach an almost quaternionic structure on M plays the same rôle as a conformal structure does in the four-dimensional theories of Donaldson and Witten. That is, we use an almost quaternionic structure γ on a $4k$-dimensional manifold solely as an auxiliary tool for introducing the notion of instantons on M and for expressing the resulting smooth invariants with the help of the associated moduli space $\mathcal{M}$ with the assurance that the final results do not depend on any particular choice of γ.

Though in the intermediate considerations we use the mathematically ill-defined path integral techniques, it is remarkable that we arrive at a mathematically rigorous and concrete recipe for computing the map

$$H^i(M^{4k},\mathbb{R}) \rightarrow H^i(\mathcal{M},\mathbb{R}), \qquad i = 1,\ldots,4k,$$

from the de Rham cohomology on M^{4k} to the de Rham cohomology on $\mathcal{M}$. This map provides a higher dimensional generalization of the well-known Donaldson map and is one of the central results of the paper. In the case when the moduli space $\mathcal{M}$ is discrete we also succeed in constructing a $4k$-dimensional analogue of the first Donaldson invariant.

Another important topic in this paper is the investigation of the topological field theory on a manifold admitting a hyperKähler structure. We construct a specific topological quantum field theory on an arbitrary $4k$-dimensional hyperKähler background and prove that in addition to the Witten-type supersymmetry this theory has a two-parameter family of new supersymmetries associated with the two-parameter family of complex structures on M^{4k}. We use the path integral technique to prove that the moduli space $\mathcal{M}$ on a hyperKähler manifold M^{4k} has a natural hypercomplex structure, and for each choice of a complex structure on M^{4k} there is a map

$$H^{r,s}(M^{4k}) \rightarrow H^{r,s}(\mathcal{M}), \qquad r,s = 1,\ldots,2k,$$

relating the Dolbeault cohomology on M^{4k} with the Dolbeault cohomology on $\mathcal{M}$. Again the recipe for calculating the Dolbeault cohomology class on $\mathcal{M}$ corresponding to any given Dolbeault cohomology class on M^{4k} is concrete and rigorous. This part of the paper provides a generalization to arbitrary $4k$ dimensions of the results obtained by Galperin and Ogievetsky [11] in the four-dimensional case.

The paper is organised as follows. In Section 2 we present a slight modification of Witten's four-dimensional topological field theory. The key idea is that the notion of self-duality absorbed by the topological field theory through some fields of the multiplet can be defined with the help of an *arbitrary* conformal structure, not necessarily coinciding with the equivalence class $g \sim \Omega^2 g$ of the Riemannian metric g which enters the field theory multiplet. The resulting version of Witten's field theory remains topological in the sense that vacuum expectation values of certain functionals are independent of both the Riemannian metric g used to define norms and the conformal structure used to define self-duality of Yang–Mills connections.

Another important observation made in this section is that a conformal structure on a four-dimensional manifold M can itself be defined in two equivalent ways. One approach is standard and consists of a specification of an equivalence class of conformally related Riemannian metrics. The other approach is to fix an isomorphism

$$\mathbb{C} \otimes \Omega^1 M \cong S \otimes \tilde{S}$$

of the complexified cotangent bundle of M with a tensor product of two rank-two complex bundles S and $\tilde{S}$ equipped with a certain real structure. However, in higher dimensions these two ways to define a conformal structure, and hence the notion of self-duality, become inequivalent to each other. It is the second approach to self-duality that survives in the category of almost quaternionic manifolds. In Section 3 we develop a topological field theory on an arbitrary $4k$-dimensional manifold admitting an almost quaternionic structure. The possibility of decomposing the bundle of two-forms on M independently of the choice of Riemannian metric becomes critically important in dimensions bigger than four. We find a specific class of functionals on the function space of the topological field theory whose vacuum expectation values are independent of both the choice of almost quaternionic structure on M^{4k} and the choice of Riemannian metric g used in calculations. The central results of this section are the construction of a generalized Donaldson map relating the de Rham cohomology on M^{4k} to the de Rham cohomology on the moduli space $\mathcal{M}$ and the calculation of the first Donaldson-like invariant.

In the final section we investigate a version of the topological field theory specially adapted to the case when the $4k$-dimensional manifold M under consideration admits a hyperKähler structure. We prove that the associated moduli space $\mathcal{M}$ has an induced hypercomplex structure, and for each choice of a complex structure on M there is a corresponding map from the Dolbeault cohomology of M to the Dolbeault cohomology of $\mathcal{M}$. This map is used to construct new invariants of hypercomplex structures.

2. A comment on Witten's topological field theory

2.1. The list of fields and the action

Let M be a four-dimensional manifold equipped with a Riemannian metric g and a principle G-bundle $\mathcal{G} \to M$ for which the structure group G is a compact Lie group. Let us specify a connection ∇ on $\mathcal{G}$ along with the collection of bosonic and fermionic fields listed in the following table:

Bosonic fields	*Fermionic fields*
$\phi \in \Gamma(M, \mathrm{ad}\,\mathcal{G})$	$\psi \in \Gamma(M, \Omega^1 M \otimes \mathrm{ad}\,\mathcal{G})$
$\lambda \in \Gamma(M, \mathrm{ad}\,\mathcal{G})$	$\eta \in \Gamma(M, \mathrm{ad}\,\mathcal{G})$
$B \in \Gamma(M, \Omega^2_+ M \otimes \mathrm{ad}\,\mathcal{G})$	$\chi \in \Gamma(M, \Omega^2_+ M \otimes \mathrm{ad}\,\mathcal{G})$

Here $\Omega^2_+ M$ denotes the bundle of self-dual two-forms on M. The action of the topological field theory on M, written in a local coordinate system $\{x^a : a = 1,2,3,4\}$, is given explicitly by [25]

$$S(g, \nabla, \phi, \lambda, B, \psi, \eta, \chi) = \int_M L \sqrt{\det g}\, dx^4, \tag{1}$$

with the Lagrangian

$$L = \mathrm{Tr}\{-\tfrac{1}{8} B_{ab} B^{ab} + \tfrac{1}{2} B_{ab} F^{ab} - \tfrac{1}{2} \nabla_a \phi \nabla^a \lambda + i \nabla_a \psi_b \chi^{ab} - i \psi^a \nabla_a \eta$$
$$-\tfrac{1}{8} i \phi [\chi_{ab}, \chi^{ab}] - \tfrac{1}{2} i \lambda [\psi_a, \psi^a] + \tfrac{1}{2} i s \phi [\eta, \eta] + \tfrac{1}{8} s\, [\phi, \lambda]^2\}, \tag{2}$$

where tangent space indices (denoted by small Latin letters) are raised with the help of the inverse metric tensor, $\nabla^a = g^{ab} \nabla_b$ etc. The symbol F_{ab} ($= \partial_a A_b - \partial_b A_a + [A_a, A_b]$) stands for the component of the curvature tensor of the connection ∇ in the chosen coordinate chart and s is an arbitrary real constant.

The functional (1) is designed to be invariant under the odd operator Q acting in the field theory function space according to the formulae [25,13]

$$QA_a = i\psi_a, \quad Q\psi_a = -\nabla_a \phi, \quad Q\phi = 0,$$
$$QB_{ab} = i[\phi, \chi_{ab}], \quad Q\chi_{ab} = B_{ab}, \quad Q\lambda = 2i\eta,$$
$$Q\eta = \tfrac{1}{2}[\phi, \lambda], \quad Qg_{ab} = 0. \tag{3}$$

The square of this operator Q^2 is exactly the infinitesimal gauge transformation with the parameter $i\phi$,

$$Q^2 A_a = -i\nabla_a \phi, \quad Q^2 \psi_a = i[\phi, \psi_a], \quad Q^2 B_{ab} = i[\phi, B_{ab}],$$
$$Q^2 \lambda = i[\phi, \lambda], \quad Q^2 \chi_{ab} = i[\phi, \chi_{ab}], \quad Q^2 \eta = i[\phi, \eta].$$

Therefore the operator Q^2 restricted to the subspace of gauge invariant functionals on the field theory function space may be identified with zero,

$$Q^2 \,|_{\text{gauge invariant functionals}} = 0,$$

and we may introduce the notion of Q-cohomology classes of gauge invariant functionals. The property of the action (1) which plays a crucial rôle in the quantum field theory interpretation of Donaldson's invariants is that the Lagrangian (2) is not just Q-closed but also Q-exact,

$$L = QV,$$

where

$$V = \mathrm{Tr}\{\tfrac{1}{2}\psi_a \nabla^a \lambda + \tfrac{1}{4} s\eta[\phi, \lambda] - \tfrac{1}{8} B_{ab}\chi^{ab} + \tfrac{1}{2} F_{ab}\chi^{ab}\}.$$

For constructing smooth invariants it is also important to show that the energy–momentum

$$T_{ab} = \frac{\delta}{\delta g^{ab}} \int_M L\sqrt{\det g}\, dx^4$$

of the theory is Q-exact. This is indeed the case, since, as one may check using (3), the operators Q and $\delta/\delta g^{ab}$ commute with each other,

$$T_{ab} = Q\frac{\delta}{\delta g^{ab}} \int_M V\sqrt{\det g}\, dx^4.$$

It is the Q-exactness of the energy–momentum tensor that is responsible for the fact that the vacuum expectation value

$$\langle \mathcal{A} \rangle = \int (\mathcal{D}A)(\mathcal{D}\phi)(\mathcal{D}\lambda)(\mathcal{D}B)(\mathcal{D}\psi)(\mathcal{D}\eta)(\mathcal{D}\chi) \times \exp\left(-\frac{1}{e^2} S(g, \nabla, \phi, \lambda, B, \psi, \eta, \chi)\right) \mathcal{A}$$

of a Q-closed and metric-independent functional $\mathcal{A}$ does not depend on the choice of a metric g used in the calculation of $\langle \mathcal{A} \rangle$ and carries only smooth information about the underlying manifold M (cf. 3.3). Witten [25] found functionals $\mathcal{A}$ which produce the Donaldson invariants [9] in this way.

In his original approach Witten [25] fixes the free parameter s to be -1 and proves that in the flat case ($M = \mathbb{R}^4$) the resulting theory coincides exactly with the twisted version of the usual $N = 2$ supersymmetric Yang–Mills theory in which the $SU(2)$ automorphism group of $N = 2$ supersymmetry gets identified with the $SU(2)_L$ subgroup of the Euclidean tangent group $SO(4) \cong SU(2)_L \times_{\mathbb{Z}_2} SU(2)_R$. Later Atiyah and Jeffrey [3] found a beautiful interpretation of the partition function $\langle 1 \rangle$ (the first Donaldson invariant) of Witten's topological field theory as the Euler number of an infinite-dimensional vector bundle over the space of gauge equivalence classes of connections. Moreover, they succeeded in reproducing Witten's action (1) with $s = 0$ from the infinite-dimensional analogue of the Mathai–Quillen form [17].

2.2. *Self-duality in four dimensions*

Self-duality is absorbed by the topological field theory through two fields, the bosonic and, respectively, fermionic $\mathrm{ad}\,\mathcal{G}$-valued two-forms B and χ. Usually the decomposition

$$\Omega^2 M = \Omega^2_+ M \oplus \Omega^2_- M, \tag{4}$$

of the bundle of two-forms into the direct sum of subbundles of self-dual $\Omega^2_+ M$ and anti-self-dual $\Omega^2_- M$ two-forms is achieved with the help of the Hodge operator $*$. If a metric g is fixed on a manifold M then the Hodge operator may be defined by the equation

$$(\omega_1, \omega_2)\,\mathrm{vol} = \omega_1 \wedge *\omega_2 \tag{5}$$

for any $\omega_1, \omega_2 \in \Omega^i M$, where $(\cdot,\cdot)$ and vol are, respectively, the scalar product on the bundle of i-forms $\Omega^i M$ and the volume form on the manifold M determined by the fixed Riemannian structure g.

The form of the definition of the Hodge operator given in (5) makes it evident that in four-dimensions the operator $*$ applied to two-forms depends only on the conformal class $[g]$ rather than on the particular representative g (multiplication of the metric g by Ω^2 multiplies the scalar product of two-forms by Ω^{-4} and the volume form by Ω^4). Thus it is a conformal rather than a Riemannian structure on M that is involved in the decomposition (4).

Usually, a conformal structure on a four-dimensional manifold M is defined as an equivalence class $[g]$ of (pseudo) Riemannian metrics on M under the relation $\hat{g} \sim g$ if $\hat{g} = \Omega^2 g$ for some nowhere vanishing function Ω. Since the Riemannian metric g is a member of the topological field theory multiplet, it is natural to define the decomposition (4) in this framework with the help of the conformal structure $[g]$ corresponding to g. It is this approach to the definition of self-duality that is used by Witten in his original work [25].

However, there exists another way to define a conformal structure on a four-dimensional manifold M. Suppose that the complexified cotangent bundle of M factors as the tensor product

$$\Omega^1_{\mathbb{C}} M \cong S \otimes \tilde{S}, \tag{6}$$

where S and $\tilde{S}$ are two complex rank-two bundles. This isomorphism specifies canonically the line subbundle $\wedge^2 S \otimes \wedge^2 \tilde{S} \subset \Omega^1_{\mathbb{C}} M \odot \Omega^1_{\mathbb{C}} M$, which in turn may be identified with the complex equivalence class of conformally related metrics. A real structure ρ on $S \oplus \tilde{S}$ determines the signature of the metrics in the conformal class [2]. If ρ provides an antilinear isomorphism between S and $\tilde{S}$, i.e.

$$(S)^{\rho} \cong \tilde{S}, \qquad (\tilde{S})^{\rho} \cong S,$$

then the conformal structure $\wedge^2 S \otimes \wedge^2 \tilde{S}$ has the Lorentzian signature. If ρ gives quaternionic structures

$$(S)^{\rho_S} \cong S, \quad {\rho_S}^2 = -1, \qquad (\tilde{S})^{\rho_{\tilde{S}}} \cong \tilde{S}, \quad {\rho_{\tilde{S}}}^2 = -1$$

on S and $\tilde{S}$ then the induced conformal structure on M has Euclidean signature. Note that if M is analytic then it is possible to view ρ as an extension of a real structure on a complexified manifold $\mathbf{M}$ whose fixed point set is M.

Since we are interested in this paper in the case of Euclidean signature, we assume from now on that vector bundles S and $\tilde{S}$ are equipped with quaternionic structures. Then the tensor products $(\otimes^p S) \otimes (\otimes^q \tilde{S})$ with $p + q$ even integers come equipped with a real structure and we adopt the notation that the symbol $(\otimes^p S) \otimes (\otimes^q \tilde{S})$ with $p + q$ even, though expressed in terms of *complex* vector bundles, stands for the corresponding *real* subbundle over the real manifold M.

Thus we conclude that a Euclidean signature conformal structure on a four-dimensional manifold M can be given as an isomorphism

$$\gamma: \Omega^1 M \to S \otimes \tilde{S}, \tag{7}$$

where S and $\tilde{S}$ are rank 2 complex vector bundles over M equipped with fibrewise quaternionic structures.

Once a conformal structure on M is specified by an isomorphism γ, the corresponding decomposition (4) becomes very transparent

$$\begin{array}{ccccc} \Omega^2 M & = & \Omega^2_+ M & \oplus & \Omega^2_- M \\ \updownarrow \gamma & & \updownarrow \gamma & & \updownarrow \gamma \\ \wedge^2 (S \otimes \tilde{S}) & = & \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}) & \oplus & (S \odot S) \otimes \wedge^2 \tilde{S} \end{array}$$

In general there is a topological obstruction, the second Stiefel–Whitney class $\epsilon \in H^1(M; \mathbb{Z}_2)$, to the global existence of vector bundles S and $\tilde{S}$ on M. However, both the definition of a conformal structure through the factoring (7) and the corresponding decomposition of $\Omega^2 M$ into self-dual and anti-self-dual

[2] Let us recall some terminology [15]. A real structure on a commutative $\mathbb{C}$-algebra A is a $\mathbb{C}$-antilinear isomorphism $A \to A$, $a \to a^{\rho}$ with the properties: $a^{\rho\rho} = a$, $(ab)^{\rho} = a^{\rho} b^{\rho}$ and $(\alpha a)^{\rho} = \bar{\alpha} a^{\rho}$ for $\alpha \in \mathbb{C}$. An extension of ρ to a real (respectively, quaternionic) structure on an A-module M is a mapping $M \to M, m \to m^{\rho}$ satisfying the conditions $(am)^{\rho} = a^{\rho} m^{\rho}, m^{\rho\rho} = m$ (respectively, $m^{\rho\rho} = -m$).

subbundles are invariant under the action of $\mathbb{Z}_2$ on $S \times \tilde{S}$ and thus rely only on the local existence of S and $\tilde{S}$. This means that such an approach to the definition of self-duality does not encounter any topological obstructions and is valid for arbitrary four-dimensional manifolds.

2.3. A version of the topological field theory in four dimensions

Suppose that in addition to the standard topological field theory structure an isomorphism $\gamma : \Omega^1 M \to S \otimes \tilde{S}$ is also specified on a four-dimensional manifold M. Then we have to choose whether to define the notion of self-duality in terms of the conformal structure $[g]$ associated with the fixed Riemannian metric g, or in terms of the decomposition $\gamma(\Omega^2 M) = \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}) \oplus (S \odot S) \otimes \wedge^2 \tilde{S}$ associated with the isomorphism γ. The first approach gives just the original formulation of Witten [25], while the second approach leads to a field theory with the action (1), (2) depending implicitly on γ through the fields B and χ. Remarkably enough, this new theory is also topological in the sense that expectation values,

$$\langle \mathcal{A} \rangle = \int (\mathcal{D}A)(\mathcal{D}\phi)(\mathcal{D}\lambda)(\mathcal{D}B)(\mathcal{D}\psi)(\mathcal{D}\eta)(\mathcal{D}\chi) \times \exp\left(-\frac{1}{e^2} S(g, \gamma, \nabla, \phi, \lambda, B, \psi, \eta, \chi)\right) \mathcal{A}$$

of Q-closed and metric- and γ-independent functionals $\mathcal{A}$ again depend neither on the choice of metric g nor on the choice of isomorphism γ used in the path integral calculations and thus encode purely smooth information about the underlying manifold M.

If the Riemannian metric g and the isomorphism γ are bound to define the same conformal structure on M, the new version of the topological field theory becomes completely equivalent to the original one [25]. However there is nothing specific in such a choice and we are free to consider g and γ as completely independent structures. Nevertheless this produces the same results as in Ref. [25]. This flexibility becomes extremely important when one attempts to generalize Witten's four-dimensional topological field theory to higher dimensions.

3. Topological field theory on almost quaternionic manifolds

3.1. Almost quaternionic structures

Let $\mathbb{H}$ denote the division ring of quaternions and $\mathbb{HP}^k$ the $4k$-dimensional quaternionic projective space. The latter may be defined either as the quotient $(\mathbb{H}^{k+1} - \{0\})/\mathbb{H}^*$, where $\mathbb{H}^*$ is the group of non-zero quaternions acting by right multiplication, or as a real submanifold of a $4k$-dimensional complex Grassmannian, $Gr(2;\mathbb{C}^{2k+2})$, of two-planes in $\mathbb{C}^{2k+2}$ singled out by a real structure ρ defined as follows. For $q \in \mathbb{H}$, we write $q = a + bj$, where $a, b \in \mathbb{C}$ and j is a unit quaternion. This decomposition provides the identification $\mathbb{H} \cong \mathbb{C}^2$ which can be readily generalised to $k+1$ quaternionic dimensions to obtain the identification $\mathbb{H}^{k+1} \cong \mathbb{C}^{2k+2}$. Let

$$\rho : \mathbb{C}^{2k+2} \to \mathbb{C}^{2k+2} \tag{8}$$

be an antilinear automorphism induced by left multiplication by j. Then $\mathbb{H}\mathrm{P}^k$ can be identified with the subset of $\mathbf{M} = Gr(2;\mathbb{C}^{2k+2})$ consisting of "real", i.e. ρ-fixed, complex two-planes in $\mathbb{C}^{2k+2}$. Such a realization of the quaternionic projective space makes transparent the canonical $GL(k,\mathbb{H})Sp(1)$-structure on $\mathbb{H}P^k$, i.e. the reduction of the structure group $GL(4k,\mathbb{R})$ of the frame bundle to $GL(k,\mathbb{H})Sp(1)$. In fact, there is a natural tautological rank 2 complex vector bundle $\tilde{S}$ on $\mathbf{M}$ whose fibre over a point $x \in \mathbf{M}$ is the two-plane in $\mathbb{C}^{2k+2}$ corresponding to x. The bundle $\tilde{S}$ is obviously a subbundle of the trivial bundle $\mathbb{C}^{2k+2} \times \mathbf{M}$, and another natural rank $2k+2$ vector bundle S^* over the Grassmannian can be defined by an exact sequence

$$0 \longrightarrow \tilde{S} \longrightarrow \mathbb{C}^{2k+2} \times \mathbf{M} \longrightarrow S^* \longrightarrow 0.$$

The cotangent bundle to the Grassmannian $\mathbf{M}$ factors as a tensor product [15]

$$\Omega^1\mathbf{M} \cong S \otimes \tilde{S},$$

where S is the dual of S^*. The quaternionic projective space $\mathbb{H}\mathrm{P}^k \subset \mathbf{M}$ inherits this important property

$$\mathbb{C} \otimes_{\mathbb{R}} \Omega^1\mathbb{H}\mathrm{P}^k = \Omega^1\mathbf{M}\,|_{\mathbb{H}\mathrm{P}^k} \cong S \otimes \tilde{S}\,|_{\mathbb{H}\mathrm{P}^k} .$$

The real structure (8) induces fibrewise quaternionic structures ρ_S and $\rho_{\tilde{S}}$ on complex vector bundles S and $\tilde{S}$ restricted to the submanifold $\mathbb{H}\mathrm{P}^k \subset \mathbf{M}$. Then the tensor product $S \otimes \tilde{S}\,|_{\mathbb{H}\mathrm{P}^k}$ comes equipped with the real structure $\rho_S \otimes \rho_{\tilde{S}}$ which singles out the cotangent bundle $\Omega^1\mathbb{H}\mathrm{P}^k$ as the corresponding real subbundle in $S \otimes \tilde{S}$. The subgroup of the structure group $GL(2k,\mathbb{C})$ of the complex bundle $S\,|_{\mathbb{H}\mathrm{P}^k}$ commuting with ρ_S is $GL(k,\mathbb{H})$ while the subgroup of the structure group $GL(2,\mathbb{C})$ of the bundle $\tilde{S}\,|_{\mathbb{H}\mathrm{P}^k}$ commuting with $\rho_{\tilde{S}}$ is $GL(1,\mathbb{H})$. This means that the quaternionic projective space inherits from the pair $(Gr(2;\mathbb{C}^4),\rho)$ a G-structure with

$$G = GL(k,\mathbb{H})Sp(1) \cong GL(k,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1).$$

Taking this canonical G-structure on $\mathbb{H}\mathrm{P}^k$ as a model, Salamon [23] introduces a category of almost quaternionic manifolds as real $4k$-dimensional manifolds admitting a $G = GL(k,\mathbb{H})Sp(1)$-structure. Locally, the reduced

$GL(k,\mathbb{H})Sp(1)$-subbundle F of the frame bundle on an almost quaternionic manifold can be lifted to a principal $GL(k,\mathbb{H}) \times Sp(1)$-bundle $\tilde{F}$ which double covers F and this enables the construction of bundles associated to representations W of $GL(k,\mathbb{H}) \times Sp(1)$. Such a bundle is defined to be $\tilde{F} \times_{GL(k,\mathbb{H})\times Sp(1)} W$, where $g \in GL(k,\mathbb{H}) \times Sp(1)$ acts on a typical element of $\tilde{F} \times W$ via $(f,w) \cdot g = (f \cdot g, g^{-1} \cdot w)$. These bundles exist globally if either $\tilde{F}$ exists globally or $(-1,-1) \in GL(k,\mathbb{H}) \times Sp(1)$ acts as the identity.

There are two basic modules we consider: S is $\mathbb{H}^k$ with $A \cdot \xi = \xi A^{-1}$, for $A \in GL(k,\mathbb{H})$ and $\xi \in \mathbb{H}^k$; and $\tilde{S}$ is $\mathbb{H}$ with $q \cdot \eta = q\eta$, for $q \in Sp(1)$ and $\eta \in \mathbb{H}$. From the realization of $GL(k,\mathbb{H})Sp(1)$ as a subgroup of $GL(4k,\mathbb{R})$, we have that the complexified cotangent bundle, $\mathbb{C} \otimes_{\mathbb{R}} \Omega^1 M$, of an almost quaternionic manifold M factors as a tensor product $S \otimes_{\mathbb{C}} \tilde{S}$. Each bundle S and $\tilde{S}$ comes equipped with natural fibrewise quaternionic structures induced, by right multiplication by a unit quaternion j, $(\xi)^\rho = -\xi j$, $\xi \in \mathbb{H}^k$, and $(\eta)^\rho = j\eta$. Then the tensor product $S \otimes_{\mathbb{C}} \tilde{S}$ admits the complex conjugation $\overline{\xi \otimes_{\mathbb{C}} \eta} = -\xi j \otimes_{\mathbb{C}} j\eta$ which singles out the cotangent bundle $\Omega^1 M$ as the corresponding real subbundle of $S \otimes_{\mathbb{C}} \tilde{S}$. It is clear that all the tensor products $(\otimes^p S) \otimes (\otimes^q \tilde{S})$ with $p + q$ even integers come equipped with a real structure and, as in Section 2, we adopt the notation that the symbol $(\otimes^p S) \otimes (\otimes^q \tilde{S})$ with $p + q$ even, though expressed in terms of *complex* vector bundles, stands for the corresponding *real* subbundle over the real manifold M.

Thus we conclude that an almost quaternionic structure on a $4k$-dimensional real manifold M is just an isomorphism

$$\gamma : \Omega^1 M \to S \otimes \tilde{S}, \tag{9}$$

where S and $\tilde{S}$ are rank $2k$ and, respectively, rank 2 complex vector bundles over M equipped with fibrewise quaternionic structures. For our purposes the most important structure this gives M is the splitting of the bundle of two-forms

$$\begin{array}{ccccc} \Omega^2 M & = & \Omega^2_+ M & \oplus & \Omega^2_- M \\ \updownarrow \gamma & & \updownarrow \gamma & & \updownarrow \gamma \\ \wedge^2 (S \otimes \tilde{S}) & = & \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}) & \oplus & (S \odot S) \otimes \wedge^2 \tilde{S} \end{array}$$

This provides a natural generalization of the notion of self-duality from four to $4k$ dimensions. Note that when $k \geq 2$ this can not be interpreted as the Hodge decomposition associated with a Riemannian metric on M.

The definition of an almost quaternionic manifold involves a reduction of the structure group and this leads to topological obstructions to the existence of such structures. In general these obstructions are not known, but in [12] it was shown that S^{4n} does not admit an almost quaternionic structure if $n > 1$. Via the work of Adams [1] this may be deduced as a special case of a result of Marchiafava [16] which states that if M is a manifold such that

$H^2(M,\mathbb{Z}/2) = 0$, $H^4(M,\mathbb{Z}) = 0$ and M does not admit an almost complex structure, then M can not admit an almost quaternionic structure.

3.2. The action and observables of the topological field theory

Let M be a compact $4k$-dimensional manifold. A topological field theory on M is defined by the following set of structures: (i) an almost quaternionic structure $\gamma: \Omega^1 M \to S\otimes\tilde{S}$; (ii) Riemannian metric g; (iii) a principle G-bundle $\mathcal{G} \to M$ equipped with a linear connection ∇; (iv) the collection of bosonic and fermionic fields on M which is formally the same as the one listed in the table of subsection 2.1. At this point the construction of the action and the odd operator Q is formally identical to the prescription given by Witten, see Eqs. (1)–(3). Witten's construction of observables can be generalised from 4 to $4k$ dimensions with the help of the following trick (cf. [6]). Consider the section $\frac{1}{2}iF + \psi + \phi \in \Gamma(M, \mathrm{ad}\,\mathcal{G} \otimes \Lambda^\bullet T^*M)$ of the Cartan algebra of the $\mathrm{ad}\,\mathcal{G}$-valued differential forms on M, where we use a differential form notation $F = F_{ab}dx^a \wedge dx^b$, $\psi = \psi_a dx^a$ It is easy to check the identity

$$(d + Q)(\tfrac{1}{2}iF + \psi + \phi) + [A, \tfrac{1}{2}iF + \psi + \phi] = 0,$$

which implies

$$(d + Q)\,\mathrm{Tr}(\tfrac{1}{2}iF + \psi + \phi)^m = 0$$

for any positive number m. Let us take $m = 2k$ (or any $m \geq 2k$) and decompose the trace

$$\mathrm{Tr}(\tfrac{1}{2}iF + \psi + \phi)^{2k} = \sum_{i=0}^{4k} \mathcal{A}_i \tag{10}$$

into a sum of homogeneous i-forms $\mathcal{A}_i$. In particular, $\mathcal{A}_0 = \mathrm{Tr}(\phi^{2k})$ and $\mathcal{A}_{4k} = \mathrm{Tr}(F^{2k})$.

By construction, the collection of differential forms $\mathcal{A}_i$, $i = 0, 1, \ldots, 4k$ satisfies the equations

$$d\mathcal{A}_i = -Q\mathcal{A}_{i+1}, \qquad i = 0, 1, \ldots, 4k-1,$$
$$d\mathcal{A}_{4k} = 0.$$

To show that the functionals $\mathcal{A} = \mathcal{A}_{i_1}\wedge\mathcal{A}_{i_2}\wedge\cdots\wedge\mathcal{A}_{i_p}$ have non-zero expectation values for some n is a more complicated task, since the calculation of $\langle\mathcal{A}_{i_1} \wedge \mathcal{A}_{i_2} \wedge \cdots \wedge \mathcal{A}_{i_p}\rangle$ relies on properties of the solution space of a very little-known non-linear differential equation. Indeed, since

$$\delta\langle\mathcal{A}\rangle/\delta(-1/e^2) = 0,$$

the expectation values $\langle\mathcal{A}\rangle$ do not depend on the coupling constant and we may calculate them in the limit $e \to 0$. In this limit the path integral is dominated by configurations on which the action is equal to zero. To describe such configurations, note that the kinetic gauge field terms in S are positive semidefinite and vanish for connections satisfying the equation

$$\mathrm{pr}^g_+(F) = 0, \tag{11}$$

where F is the curvature two-form and pr^g_+ is the orthogonal (relative to the metric g) projection from the space $\Omega^2 M$ to its subspace $\gamma^{-1}(\wedge^2 S \otimes (\tilde{S} \odot \tilde{S}))$. Since the metric g is arbitrary, we can always choose it in such a way that the decomposition

$$\Omega^2 M = \gamma^{-1}\left(\wedge^2 S \otimes (\tilde{S} \odot \tilde{S})\right) \oplus \gamma^{-1}\left((S \odot S) \otimes \wedge^2 \tilde{S}\right)$$

is orthogonal relative to g (e.g., $g = \gamma^{-1}(\epsilon \otimes \tilde{\epsilon})$, where ϵ and $\tilde{\epsilon}$ are any nowhere vanishing sections of $\wedge^2 S$ and $\wedge^2 \tilde{S}$ respectively). Then Eq. (11) takes the form

$$\gamma_+(F) = 0, \tag{12}$$

where γ_+ is given by the composition

$$\gamma_+ : \Omega^2 M \xrightarrow{\gamma} \wedge^2(S \otimes \tilde{S}) \xrightarrow{\mathrm{pr}} \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}).$$

This equation states that path integrals $\langle \mathcal{A}_{i_1} \wedge \mathcal{A}_{i_2} \wedge \cdots \wedge \mathcal{A}_{i_p} \rangle$ are concentrated on the superspace containing the moduli space of *instantons*, that is to say connections ∇ whose curvature lies in $\gamma^{-1}\left((S \odot S) \otimes \wedge^2 \tilde{S}\right)$. Fortunately Eq. (12) is not a completely unknown one. In 3.3 and 3.4 below we give a summary of known properties of the moduli spaces of solutions of Eq. (12) and then proceed further with the investigation of the proposed invariants of the type $\langle \mathcal{A}_{i_1} \wedge \mathcal{A}_{i_2} \wedge \cdots \wedge \mathcal{A}_{i_p} \rangle$.

3.3. Quaternionic Kähler structure

Starting with the works by Mamone Capria and Salamon [14] and Nitta [18] a number of mathematicians have paid attention to the structure of moduli spaces of solutions of Eq. (12) in the case when the underlying almost quaternionic manifold M admits a quaternionic Kähler metric. Let us recall some basic notions [14]. Let M be a compact $4k$-dimensional manifold, $k \geq 2$, equipped with an almost quaternionic structure $\gamma : \Omega^1 M \to S \otimes \tilde{S}$. A Riemannian metric g on M is said to be *compatible* with the almost quaternionic structure if it can be represented in the form $g = \gamma^{-1}(\epsilon \otimes \tilde{\epsilon})$ for some symplectic forms ϵ and $\tilde{\epsilon}$ on S and $\tilde{S}$ respectively.

A *quaternionic Kähler structure* [21,4] on M is an almost quaternionic structure γ together with a compatible metric $g = \gamma^{-1}(\epsilon \otimes \tilde{\epsilon})$ and connections ∇ on S and $\tilde{S}$ such that symplectic forms are horizontal,

$$\nabla \epsilon = \nabla \tilde{\epsilon} = 0, \tag{13}$$

and the induced connection ∇ on $\Omega^1 M$ is the Levi-Civita connection corresponding to g. This is the same as saying that a quaternionic Kähler manifold is a Riemannian manifold whose holonomy group lies in $Sp(k)Sp(1)$. Examples of such manifolds are provided by quaternionic projective spaces $\mathbb{HP}^k$.

Let us consider the group $\mathcal{G}$ consisting of those automorphisms of the bundle $\tilde{S}$ which preserve the symplectic form $\tilde{\epsilon}$ and commute with the fibrewise quaternionic structure $\rho: \tilde{S} \to \tilde{S}$. Fibrewise, the action of $\mathcal{G}$ can be viewed as left multiplication by $Sp(1)$. Using the natural inclusion

$$\tilde{S} \odot \tilde{S} \hookrightarrow \tilde{S} \otimes \tilde{S} \cong_{\tilde{\epsilon}} \tilde{S} \otimes \tilde{S}^* = \operatorname{End} \tilde{S}$$

one can identify the Lie algebra of $Sp(1)$ with $\tilde{S} \odot \tilde{S}$. With this identification the action of $\tilde{S} \odot \tilde{S}$ on the tangent bundle $TM \cong S^* \otimes \tilde{S}^*$ is given by the composition

$$TM \otimes (\tilde{S} \odot \tilde{S}) \hookrightarrow S^* \otimes \tilde{S}^* \otimes \tilde{S} \otimes \tilde{S} \stackrel{\mathrm{id} \otimes \mathrm{trace} \otimes \tilde{\epsilon}}{\longrightarrow} S^* \otimes \tilde{S}^* \cong TM.$$

Moreover, if $J, K \in \tilde{S} \odot \tilde{S}$, then as endomorphisms of TM ,

$$JK + KJ = -\langle J, K \rangle \, \mathrm{id} \tag{14}$$

where $\langle \, , \, \rangle$ is the fibrewise scalar product induced on $\tilde{S} \odot \tilde{S}$ by the symplectic form $\tilde{\epsilon}$.

Since fibrewise the Lie algebra of $\mathcal{G}$ acts on TM as the left multiplication by an imaginary quaternion from $sp(1)$, the bundle $\tilde{S} \odot \tilde{S} \subset \operatorname{End} TM$ has a local (but not in general global) basis $\{I, J, K\}$ of endomorphisms satisfying the familiar identities

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K \tag{15}$$

3.4. Moduli spaces of instantons

A general theory for instantons on quaternionic Kähler manifolds has been developed by Mamone-Capria and Salamon [14] and by Nitta [18–20]: Let V be complex unitary vector bundle on the quaternionic Kähler manifold M with a unitary connection ∇. Then ∇ is an *instanton* if its curvature F_∇ is in $\gamma^{-1}\left((S \odot S) \otimes \Lambda^2 \tilde{S} \otimes \operatorname{End} V\right)$. Such a connection is a solution to the Yang–Mills equations. The pull-back of the bundle to the twistor space Z of M is a holomorphic bundle when ∇ is an instanton. If the scalar curvature of M is positive then Z is Kähler and the pulled back connection is Hermitian-Einstein. Indeed, the Atiyah–Ward correspondence has been generalized to $4k$-dimensions [18,22,5].

The structure of the corresponding moduli-space turns out to be much more complicated than in four-dimensions, where the moduli space is a manifold for generic metrics and the dimension is known [10,2]. However, recently some progress has been made using results from algebraic geometry on null correlation bundles [20,7,8]: consider the homogeneous vector bundle S on the symmetric space $\mathbb{H}P^k$. Instantons on S are called *one-instantons*. The unique irreducible connection ∇ on S is an instanton and is called *the standard one-instanton*. The action of $SL(k+1,\mathbb{H})$ on $\mathbb{H}P^k$ preserves the $SL(k,\mathbb{H})Sp(1)$ structure so $g^*\nabla$ is an instanton on the bundle g^*S. Using a $Sp(k)$-bundle isomorphism $g^*S \xrightarrow{\sim} S$ we obtain a 1-instanton $g^*\nabla$ on S which is unique up to $Sp(k)$-gauge transformations on S. Then the moduli space $\mathcal{M}_k$ of 1-instantons on $\mathbb{H}P^k$ is identified with the $k(2k+3)$-dimensional homogeneous space $SL(k+1,\mathbb{H})/Sp(k+1,\mathbb{H})$ via the correspondence $g \to g^*\nabla$. Furthermore, $\mathcal{M}_k$ has a natural compactification $\hat{\mathcal{M}}_k$. Thus good moduli spaces do exist also for $k > 1$.

The bundle S on any quaternionic Kähler manifold M always has instantons but not much is known about the moduli space. As in dimension four there is an elliptic complex [19] where the first cohomology group H^1 is (or contains) the tangent space at an irreducible instanton. But there are problems with vanishing theorems and whether H^1 represents the tangent space. If M is one of the Grassmannians $Gr(2;\mathbb{C}^m)$, $Gr(4;\mathbb{R}^m)$, then $H^1 = 0$ and the moduli space should be discrete.

3.5. The generalized Donaldson map

Motivated by the results discussed in the preceding subsection, we proceed with the topological field theory on an almost quaternionic manifold M under the assumption that the corresponding moduli space of irreducible connections satisfying the Eq. (12) is a finite dimensional smooth manifold $\mathcal{M}$ and, for an irreducible connection ∇, the solution space

$$\mathbf{H}^2 = \{\chi \in \Gamma(M, \Omega^2_+ M \otimes \mathrm{ad}\,\mathcal{G}) \mid \nabla^a \chi_{ab} = 0\} \tag{16}$$

is zero. Note that Eq. (16) is precisely the dynamical equation for the field χ from the topological field theory multiplet calculated in the small coupling constant approximation (which is viable since the expectation values $\langle \mathcal{A}_{i_1} \wedge \mathcal{A}_{i_2} \wedge \cdots \wedge \mathcal{A}_{i_p} \rangle$ do not depend on a choice of e at all).

Since the requirement of irreducibility implies the vanishing of the space

$$\mathbf{H}^1 = \{\phi \in \Gamma(M, \mathrm{ad}\,\mathcal{G}) \mid \nabla\phi = 0\},$$

we conclude that path integrals $\langle \mathcal{A}_{i_1} \wedge \mathcal{A}_{i_2} \wedge \cdots \wedge \mathcal{A}_{i_p} \rangle$ are concentrated on the superspace, $\widetilde{\mathcal{M}}$, of all configurations $(\nabla, \psi, \phi, B, \chi, \lambda, \eta)$ satisfying the equations

$$\phi = \eta = \lambda = \chi = 0, \tag{17}$$

$$\gamma_+(F) = 0, \tag{18}$$

$$\gamma_+(\nabla\psi) = 0, \quad \nabla * \psi = 0, \tag{19}$$

where $*$ stands for the Hodge operator $*: \Omega^i M \to \Omega^{4k-i} M$ associated with the metric g. Note that such configurations are Q-invariant, and the Eqs. (19) coincide exactly with equations describing infinitesimal deformations of solutions to Eq. (18). Put another way, solutions ψ of Eq. (18) describe tangent vectors (with assigned odd parity) to the moduli space $\mathcal{M}$ of instantons. Thus the superspace $\widetilde{\mathcal{M}}$ is just the tangent space to $\mathcal{M}$ with fibres being assigned odd parity. Having made this identification, we immediately recover the meaning of the operator Q—the formula $QA_a = i\psi_a$ shows that the operator Q reduced to the space of configurations satisfying (17)–(19) is exactly the exterior derivative of $\mathcal{M}$. Note also that due to the odd character of zero modes of ψ a general function $f(A, \psi)$ on the superspace $\widetilde{\mathcal{M}}$ can be represented as a finite series

$$f(a_i, \psi^j) = f^{[0]}(a_i) + f_j^{[1]}(a_i)\psi^j + \cdots + f_{j_1 \ldots j_n}^{[n]}(a_i)\psi^{j_1} \cdots \psi^{j_n}, \qquad (20)$$

where a_i, $i = 1, 2, \ldots, n = \dim \mathcal{M}$, is a local coordinate system on $\mathcal{M}$. Thus f can be interpreted as a collection $\{f^{[0]}, f^{[1]}, \ldots, f^{[n]}\}$ of differential forms on the moduli space $\mathcal{M}$. Let Π be the parity change functor [15], then the superspace $\widetilde{\mathcal{M}} = \Pi T\mathcal{M}$ has a distinguished volume form, vol $= da_1 \cdots da_n d\psi^1 \cdots d\psi^n$. Thus the quantity $\int_{\widetilde{\mathcal{M}}} f$ vol is well-defined and is equal to the usual integral of the highest rank component $f^{[n]}$ of f over the moduli space

$$\int_{\widetilde{\mathcal{M}}} f \, \mathrm{vol} = \int_{\mathcal{M}} f^{[n]}.$$

Note that lower rank components of f do not contribute to this integral.

Now let ω be an arbitrary differential i-form on M not depending on the fields of the topological field theory. One easily checks that the observable $\int_M \mathcal{A}_{4k-i} \wedge \omega$ is Q-closed,

$$Q \int_M \mathcal{A}_{4k-i} \wedge \omega = \int_M Q\mathcal{A}_{4k-i} \wedge \omega = -\int_M d(\mathcal{A}_{4k-i-1} \wedge \omega) = 0.$$

Therefore the functional

$$\tilde{\omega} = \int_M \mathcal{A}_{4k-i} \wedge \omega$$

satisfies the requirements needed for constructing smooth invariants.

It is straightforward to show that the recipe proposed by Witten [25] for calculating the four-dimensional Donaldson invariants survives in the $4k$-dimensional almost quaternionic framework. To calculate $\langle \int_M \mathcal{A}_{4k-i} \wedge \omega \rangle$ one proceeds as follows:

(a) each time the connection ∇ appears in the functional $\mathcal{A}_{4k-i}$ (which, we recall, is defined by the decomposition (10)), replace it by the classical instanton configuration;
(b) each time ψ appears in $\mathcal{A}_{4k-i}$ replace it by the expression $\psi = \sum_{j=1}^{n} \psi^j \times u_{(j)}(x)$, where $\{u_{(1)}, \dots, u_{(n)}\}$ is a basis of the solution space of Eqs. (19) on the chosen instanton background and ψ^j are the fermion zero mode coordinates that appear in (20);
(c) each time ϕ appears in $\mathcal{A}_{4k-i}$ replace it by the solution $\langle\phi\rangle$ of the equation

$$\nabla^a \nabla_a \langle\phi\rangle = i[\psi^a, \psi_a]$$

with ψ replaced by its zero modes.

Let us denote by $\hat{\mathcal{A}}_{4k-i}$ the result of such substitution. Then, according to Witten's arguments [25], we have

$$\left\langle \int_M \mathcal{A}_{4k-i} \wedge \omega \right\rangle = \int_{\mathcal{M}} \Phi_\omega \,\mathrm{vol},$$

where

$$\Phi_\omega = \int_M \hat{\mathcal{A}}_{4k-i} \wedge \omega. \tag{21}$$

From Eqs. (10) and (21) it follows that Φ_ω is a function on the moduli space $\mathcal{M}$ which is homogeneous of degree i in ψ. According to the decomposition (20), this implies that Φ_ω is a differential form on $\mathcal{M}$ of the same degree as ω. Moreover, if $\omega = d\sigma$ for some $(i-1)$-form σ then Φ_ω is zero, since the functional

$$\int_M \mathcal{A}_{4k-i} \wedge d\sigma = Q\left((-1)^i \int_M \mathcal{A}_{4k-i+1} \wedge \sigma \right)$$

is Q-exact. Thus the topological quantum field theory on a $4k$-manifold M admitting an almost quaternionic structure gives a map

$$\begin{aligned} H^i(M^{4k}, \mathbb{R}) &\to H^i(\mathcal{M}, \mathbb{R}) \\ \omega &\mapsto \Phi_\omega \end{aligned} \tag{22}$$

from the de Rham cohomology of M to the de Rham cohomology of $\mathcal{M}$. In four dimensions this is the well-known Donaldson map [25,11].

It is clear that if the rank of ω is not equal to the dimension of the moduli space, the expectation value $\langle\tilde{\omega}\rangle$ is zero. To obtain a non-trivial result, one should take a collection of non-zero cohomology classes, $\omega_1 \in H^{s_1}(M, \mathbb{R}), \dots, \omega_p \in H^{s_p}(M, \mathbb{R})$, so that $s_1 + \cdots + s_p = \dim \mathcal{M}$. Then the expectation value

$$\langle\tilde{\omega}_1 \cdots \tilde{\omega}_p\rangle = \int_{\mathcal{M}} \Phi_{\omega_1} \wedge \cdots \wedge \Phi_{\omega_p}$$

is an integral of a n-form on $\mathcal{M}$. We stress once more that the quantities of the form $\langle\tilde{\omega}_1 \cdots \tilde{\omega}_p\rangle$ do not depend on the choice of almost quaternionic structure used in calculations and thus encode only smooth information. They provide a higher dimensional generalization of the four dimensional Donaldson invariants.

Concluding this subsection we note that though the quantum field theory has not achieved yet the status of a mathematically rigorous model, we succeeded in extracting from it a *concrete* recipe for calculating the image of any cohomology class $[\omega]$ on M under the Donaldson-like map (22)—take any representative ω of a given cohomology class on M and calculate the integral (21) resulting in a differential form Φ_ω on $\mathcal{M}$ with the same degree as ω. It is only in the proof that the result is actually a *closed* differential form whose cohomology class $[\Phi_\omega]$ does *not* depend on a choice of a particular representative of $[\omega]$ that quantum field theory plays a rôle.

3.6. The partition function and the first invariant

Let M be a compact quaternionic Kähler manifold and suppose the moduli space $\mathcal{M}$ of solutions to (12) is discrete. Also assume there are no zero-modes of any other fields in the topological field theory multiplet (configurations $A \neq 0$ and $\gamma_+(F) = 0$, $\psi = 0$, $B = 0$, $\chi = 0$, $\eta = 0$, $\phi = 0$, $\lambda = 0$ are Q-invariant). In this case the partition function,

$$Z = \int \mathcal{D}X e^{-S(X)}$$

will in general be non-zero. The arguments formally identical to the ones used by Witten give the following result

$$Z = \sum_{\text{instantons}} \pm 1,$$

which is a smooth invariant of a $4k$-dimensional manifold under investigation. It provides a higher dimensional analogue of the first Donaldson invariant.

4. Topological field theory on a hyperKähler manifold

4.1. Hypercomplex and hyperKähler structures

A *hyperKähler structure* [23,4] on a $4k$-dimensional manifold M is a quaternionic Kähler structure with the property that the canonical connection (13) is flat on the bundle $\tilde{S}$. This is the same as saying that the holonomy of M lies in $Sp(k)$. Since the bundle $\tilde{S} \odot \tilde{S} \subset \operatorname{End} TM$ over a hyperKähler manifold is globally trivial, it has now a *global* basis $\{I, J, K\}$ of endomorphisms satisfying relations (15). Moreover, the almost complex structures I, J and K can be chosen to be horizontal relative to the canonical connection ∇. The latter condition implies that they are integrable. Therefore a hyperKähler manifold is a particular example of a *hypercomplex* manifold, which by definition is a manifold admitting three *complex* structures I, J and K such that $IJ = -JI = K$.

4.2. The list of fields and the action of the topological field theory

Let M be a compact $4k$-dimensional hyperKähler manifold. A topological field theory on M is defined by the following set of structures

(a) a principle G-bundle, $\mathcal{G} \to M$, equipped with a linear connection ∇;

(b) the collection of bosonic and fermionic fields on M listed in the following table:

Bosonic fields	*Fermionic fields*
$\phi \in \Gamma(M, \operatorname{ad}\mathcal{G})$	$\psi \in \Gamma(M, \Omega^1 M \otimes \operatorname{ad}\mathcal{G})$
$\lambda \in \Gamma(M, \wedge^2 S \otimes \operatorname{ad}\mathcal{G})$	$\eta \in \Gamma(M, \wedge^2 S \otimes \operatorname{ad}\mathcal{G})$
$B \in \Gamma(M, \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}) \otimes \operatorname{ad}\mathcal{G})$	$\chi \in \Gamma(M, \wedge^2 S \otimes (\tilde{S} \odot \tilde{S}) \otimes \operatorname{ad}\mathcal{G})$

Note the difference between the topological field theory multiplet on a hyperKähler manifold and the ones discussed in the preceding sections—the fields λ and η are sections of the bundle $\wedge^2 S \otimes \operatorname{ad}\mathcal{G}$ rather than $\operatorname{ad}\mathcal{G}$.

Again we find it more economical to describe the topological field theory in a local coordinate chart $\{x^a : a = 1, \ldots, 4k\}$ on M and local frames, $\{e^A : A = 1, \ldots, 2k\}$ and $\{e^{\dot{A}} : \dot{A} = 1, 2\}$, of the bundles S and $\tilde{S}$ respectively, though of course nothing depends on these particular choices. Moreover, this time it is very convenient to use the underlying almost quaternionic structure

$$\gamma^{-1}(e^A \otimes e^{\dot{A}}) = \gamma_a^{A\dot{A}}\, dx^a$$

to transform all fields of the multiplet into their "spinor" forms:

$$g_{ab} = \gamma_a^{A\dot{A}} \gamma_a^{B\dot{B}} \epsilon_{AB} \tilde{\epsilon}_{\dot{A}\dot{B}}, \quad \nabla_a = \gamma_a^{A\dot{A}} \nabla_{A\dot{A}}, \quad \psi_a = \gamma_a^{A\dot{A}} \psi_{A\dot{A}}$$

$$F_{ab} = \gamma_a^{A\dot{A}} \gamma_a^{B\dot{B}} (F^+_{AB\dot{A}\dot{B}} + F^-_{AB} \tilde{\epsilon}_{\dot{A}\dot{B}})$$

where $\tilde{\epsilon}_{\dot{A}\dot{B}}$ is the component of the symplectic form, and

$$F^+_{AB\dot{A}\dot{B}} = F^+_{[AB](\dot{A}\dot{B})}, \qquad F^-_{AB} = F^-_{(AB)}$$

are the two components of the curvature tensor.

As in 2.1 and 3.2, we next define the odd operator Q acting in the field theory function space according to the formulae

$$QA_{A\dot{A}} = i\psi_{A\dot{A}}, \quad Q\psi_{A\dot{A}} = -\nabla_{A\dot{A}}\phi, \quad Q\phi = 0,$$
$$QB_{AB\dot{A}\dot{B}} = i[\phi, \chi_{AB\dot{A}\dot{B}}], \quad Q\chi_{AB\dot{A}\dot{B}} = B_{AB\dot{A}\dot{B}}, \quad Q\lambda_{AB} = 2i\eta_{AB},$$
$$Q\eta_{AB} = \tfrac{1}{2}[\phi, \lambda_{AB}], \quad Q\gamma_a^{A\dot{A}} = 0. \tag{23}$$

The square of this operator,

$$Q^2 = \delta_{i\phi},$$

is exactly the infinitesimal gauge transformation $\delta_{i\phi}$ with the parameter $i\phi$, and thus

$$Q^2 \,|_{\text{gauge invariant functionals}} = 0.$$

Let us consider the functional $\int_M V\sqrt{\det g}\, dx^{4k}$ with

$$V = -\,\mathrm{Tr}\{\tfrac{1}{2}\lambda^{AB}\nabla_{A\dot{C}}\psi_B{}^{\dot{C}} + \tfrac{1}{4}\eta_{AB}[\phi, \lambda^{AB}] + \tfrac{1}{8}B_{AB\dot{A}\dot{B}}\chi^{AB\dot{A}\dot{B}} - \tfrac{1}{2}F^{+}_{AB\dot{A}\dot{B}}\chi^{AB\dot{A}\dot{B}}\}. \tag{24}$$

Since V is gauge invariant, we immediately conclude that the action

$$S(\gamma, \nabla, \phi, \lambda, B, \psi, \eta, \chi) = \int_M L\sqrt{\det g}\, dx^{4k}, \tag{25}$$

with the Lagrangian

$$\begin{aligned} L &= QV \\ &= \mathrm{Tr}\{-\tfrac{1}{8}B_{AB\dot{A}\dot{B}}B^{AB\dot{A}\dot{B}} + \tfrac{1}{2}F^{+}_{AB\dot{A}\dot{B}}B^{AB\dot{A}\dot{B}} - \tfrac{1}{2}\nabla_{A\dot{C}}\phi\nabla_B{}^{\dot{C}}\lambda^{AB} \\ &\quad + i\nabla_{A\dot{A}}\psi_{B\dot{B}}\chi^{AB\dot{A}\dot{B}} + i\nabla_{A\dot{C}}\psi_B{}^{\dot{C}}\eta^{AB} - \tfrac{1}{8}i\phi[\chi_{AB\dot{A}\dot{B}}, \chi^{AB\dot{A}\dot{B}}] \\ &\quad - \tfrac{1}{2}i\lambda^{AB}[\psi_{A\dot{C}}, \psi_B{}^{\dot{C}}] - \tfrac{1}{2}i\phi[\eta_{AB}, \eta^{AB}] - \tfrac{1}{8}[\phi, \lambda_{AB}][\phi, \lambda^{AB}]\}, \end{aligned} \tag{26}$$

is invariant under supersymmetric Q-transformations (23).

4.3. *Extended supersymmetry*

In this subsection we investigate the possibility of extended supersymmetry in the $4k$-dimensional topological field theory. The result is that in the case when the underlying manifold M admits a hyperKähler structure the topological field theory does indeed possess three additional supersymmetries which induce three complex structures on the associated instanton moduli space.

Recall that the bundle $\tilde{S}\odot\tilde{S} \subset \mathrm{End}\, TM$ can be viewed as a coefficient bundle of imaginary quaternions acting on the tangent bundle at each point as if by right multiplication. Let J be any global section of $\tilde{S} \odot \tilde{S}$. Let us define the odd operator $Q_J = J^{\dot{A}\dot{B}} Q_{\dot{A}\dot{B}}$ acting in the function space of the topological field theory according to the formulae

$$
\begin{aligned}
&Q_J A_{A\dot{A}} = iJ_{\dot{A}}{}^{\dot{B}} \psi_{A\dot{B}}, \quad Q_J \psi_{A\dot{A}} = J_{\dot{A}}{}^{\dot{B}} \nabla_{A\dot{B}} \phi, \quad Q_J \phi = 0, \\
&Q_J \lambda_{AB} = \tfrac{1}{2} i \chi_{AB\dot{A}\dot{B}} J^{\dot{A}\dot{B}}, \quad Q_J \eta_{AB} = \tfrac{1}{4} B_{AB\dot{A}\dot{B}} J^{\dot{A}\dot{B}}, \quad Q_J \gamma_a^{A\dot{A}} = 0, \\
&Q_J B_{AB\dot{A}\dot{B}} = \mathrm{Sym}_{\dot{A}\dot{B}} \left\{ J_{\dot{A}}{}^{\dot{C}} \left(i[\phi, \chi_{AB\dot{C}\dot{B}}] - 4QF^{+}_{AB\dot{C}\dot{B}} \right) + 2iJ_{\dot{A}\dot{B}}[\phi, \eta_{AB}] \right\}, \\
&Q_J \chi_{AB\dot{A}\dot{B}} = \mathrm{Sym}_{\dot{A}\dot{B}} \left\{ J_{\dot{A}}{}^{\dot{C}} \left(-B_{AB\dot{C}\dot{B}} + 4F^{+}_{AB\dot{C}\dot{B}} \right) + J_{\dot{A}\dot{B}}[\phi, \eta_{AB}] \right\},
\end{aligned} \tag{27}
$$

where $\mathrm{Sym}_{\dot{A}\dot{B}}$ denotes symmetrisation over indices $\dot{A}$ and $\dot{B}$ and the symplectic form $\tilde{\epsilon}_{\dot{A}\dot{B}}$ and its inverse, $\tilde{\epsilon}^{\dot{A}\dot{B}}$,

$$\tilde{\epsilon}_{\dot{A}\dot{C}} \tilde{\epsilon}^{\dot{B}\dot{C}} = \delta_{\dot{A}}{}^{\dot{B}},$$

are used to raise and lower indices, $\tilde{\epsilon}^{\dot{B}\dot{C}} J_{\dot{A}\dot{C}} = J_{\dot{A}}{}^{\dot{B}}$. These transformation laws are inspired by the algebraic structure of the extended supersymmetry in $N = 2$ Yang–Mills theory [11].

Straightforward but tedious calculations show that for *any* global sections J and K of $\tilde{S} \odot \tilde{S}$ the corresponding operators satisfy the commutation relations

$$
\begin{aligned}
&Q_J Q_K + Q_K Q_J = \langle J, K \rangle \delta_{i\phi}, \\
&\quad QQ_J + Q_J Q = 0,
\end{aligned}
$$

which imply

$$
\begin{aligned}
&Q_J Q_K + Q_K Q_J \,|_{\text{gauge invariant functionals}} = 0, \\
&\quad QQ_J + Q_J Q \,|_{\text{gauge invariant functionals}} = 0.
\end{aligned} \tag{28}
$$

Let us now consider a topological field theory on a quaternionic Kähler manifold M with the action (25). Again it is straightforward to prove that this action is invariant under the supersymmetry transformations (27) $Q_J S = 0$ if and only if the section J of $\tilde{S} \odot \tilde{S}$ is *horizontal* relative to the canonical connection (13). But the curvature tensor R of M regarded as a self-adjoint endomorphism of $\Lambda^2 \Omega^1 M$ has the restriction [24]

$$R \,|_{\tilde{S} \odot \tilde{S}} = \Lambda \, \mathrm{id}_{\tilde{S} \odot \tilde{S}},$$

where Λ is a positive multiple of the scalar curvature of M. Therefore, the bundle $\tilde{S} \odot \tilde{S}$ admits a global horizontal section if and only if the scalar curvature vanishes which implies the flatness of $\tilde{S} \odot \tilde{S}$. If M is connected this in turn implies that $\tilde{S} \odot \tilde{S}$ is trivial and M is hyperKähler. Therefore we conclude that the topological field theory on a hyperKähler manifold possesses three additional supersymmetries $Q_{\dot{A}\dot{B}}$ described by transformation laws (27).

Under the assumptions that the moduli space $\mathcal{M}$ of irreducible instantons on an almost quaternionic manifold is finite dimensional and the second cohomology group (16) vanishes we identified in 4.2 the supersymmetry operator Q with the exterior derivative d of $\mathcal{M}$. Now we see that under the same assumptions, for any complex structure J on a hyperKähler manifold M there exist complex conjugate operators

$$q = \tfrac{1}{2}(Q + iQ_J), \qquad \bar{q} = \tfrac{1}{2}(Q - iQ_J),$$

which play the rôle of ∂ and $\overline{\partial}$ in complex analysis. According to (28), they satisfy commutation algebra

$$q^2 = 0, \quad \bar{q}^2 = 0, \quad q\bar{q} + \bar{q}q = 0,$$

and thus induce a complex structure on the moduli space $\mathcal{M}$. This observation clarifies the geometric meaning of operators Q_J—they transfer the hypercomplex structure from a hyperKähler manifold M to the corresponding moduli space $\mathcal{M}$—and provides a higher dimensional generalization of the result by Galperin and Ogievetsky [11] who also used the path integral techniques to show that the moduli space of instantons on a four-dimensional hyperKähler manifold has an induced hypercomplex structure.

4.4. New invariants of hypercomplex structures

The Q_J-invariance of the action (25) for horizontal sections J of $\tilde{S} \odot \tilde{S}$ becomes transparent when we note that the Lagrangian (26) on a hyperKähler manifold can be represented in the form

$$L = -\tfrac{1}{24} Q Q^{\dot{A}\dot{B}} Q_{\dot{A}\dot{C}} Q_{\dot{B}}{}^{\dot{C}} \operatorname{Tr}(\lambda_{AB}\lambda^{AB}). \tag{29}$$

This representation plays a crucial rôle in our subsequent discussion, since it implies, after integrating by parts as in Ref. [25], that the expectation value

$$\langle \mathcal{A} \rangle = \int \mathcal{D}X e^{-S(X)} \mathcal{A}(X)$$

of any functional $\mathcal{A}$ vanishing under an arbitrary combination of Q and $Q_{\dot{A}\dot{B}}$ is zero. In particular,

$$\langle (cQ + c^{\dot{A}\dot{B}} Q_{\dot{A}\dot{B}})\mathcal{A} \rangle \equiv 0 \tag{30}$$

for any parameters c, $c^{\dot{A}\dot{B}}$ and functional $\mathcal{A}$.

Let us fix a complex structure J on a hyperKähler manifold M. Suppose that we have found a set of differential forms $\mathcal{A}_{r,s}$, $r, s = 1, \dots, 2k$, which have the type (r, s) relative to the chosen complex structure and satisfy the equations

$$\overline{\partial}\mathcal{A}_{r,s} = -\bar{q}\mathcal{A}_{r,s+1}. \tag{31}$$

If $\omega_{r,s}$ is an arbitrary $\overline{\partial}$-closed differential (r,s)-form on M which does not depend on fields of the topological field theory, then the observable

$$\tilde{\omega} = \int_M \mathcal{A}_{2k-r,2k-s} \wedge \omega_{r,s}$$

is a $\bar{q}$-closed functional on the function space of the topological field theory

$$\bar{q} \int_M \mathcal{A}_{2k-r,2k-s} \wedge \omega_{r,s} = \int_M \bar{q}\mathcal{A}_{2k-r,2k-s} \wedge \omega_{r,s}$$
$$= -\int_M \overline{\partial}(\mathcal{A}_{2k-r,2k-s-1} \wedge \omega) = 0.$$

Therefore the expectation value $\langle\tilde{\omega}\rangle$ for any $\overline{\partial}$-closed (r,s)-form on M encodes the information only of a fixed hypercomplex structure on M and does not depend on a choice of a particular hyperKähler metric used in calculations. Since this expectation value is also independent of the coupling constant e, one concludes that to calculate $\langle\tilde{\omega}\rangle$ in the limit $e \to 0$ it is enough to know the auxiliary functionals $\mathcal{A}_{r,s}$, $r,s = 1,\ldots,2k$, only on the mass shell, when the fields of the topological field theory multiplet satisfy classical equations of motion.

We can construct a system of functionals $\mathcal{A}_{r,s}$, $r,s = 1,\ldots,2k$ satisfying (31) on the mass shell as follows. Let us decompose the curvature two-form F and the fermionic 1-form ψ into the sum

$$F = F^{2,0} + F^{1,1} + F^{0,2}, \quad \psi = \psi^{1,0} + \psi^{0,1}$$

of form of definite (r,s) type relative to the chosen complex structure J. Then, since

$$F^{0,2} = 0,\ F^{2,0} = 0, \quad \nabla^{0,1}\psi^{0,1} = 0, \quad \nabla^{1,0}\psi^{1,0} = 0$$

on the mass shell, one finds that

$$(\overline{\partial}+\bar{q})(\tfrac{1}{2}iF^{1,1}-\psi^{1,0}+\psi^{0,1}+\phi)+[A,\tfrac{1}{2}iF^{1,1}-\psi^{1,0}+\psi^{0,1}+\phi]\,|_{\text{mass shell}} = 0,$$

which implies

$$(\overline{\partial} + \bar{q})\,\mathrm{Tr}(\tfrac{1}{2}iF^{1,1} - \psi^{1,0} + \psi^{0,1} + \phi)^m\,|_{\text{mass shell}} = 0$$

for any positive integer m. Let us take $m = 2k$ (or any $m \geq 2k$) and decompose the trace

$$\mathrm{Tr}(\tfrac{1}{2}iF^{1,1} - \psi^{1,0} + \psi^{0,1} + \phi)^{2k} = \sum_{r=0}^{2k}\sum_{s=0}^{2k} \mathcal{A}_{r,s}$$

into a sum of (r,s)-forms $\mathcal{A}_i$. In particular, $\mathcal{A}_0 = \mathrm{Tr}(\phi^{2k})$ and $\mathcal{A}_{2k,2k} = \mathrm{Tr}(F^{1,1})^{2k}$.

By construction, the collection of differential forms $\mathcal{A}_{r,s}$, $r,s = 1,\ldots,2k$, satisfies the required Eqs. (31) on the mass shell.

Now, as in 4.3, we have

$$\langle\tilde{\omega}\rangle = \left\langle \int_M \mathcal{A}_{2k-r,2k-s} \wedge \omega_{r,s} \right\rangle = \int_{\mathcal{M}} \Phi_\omega^{r,s}\,\mathrm{vol},$$

where

$$\Phi_\omega^{r,s} = \int_M \hat{\mathcal{A}}_{2k-r,2k-s} \wedge \omega_{r,s} \tag{32}$$

and $\hat{\mathcal{A}}_{2k-r,2k-s}$ are obtained from the functionals $\mathcal{A}_{2k-r,2k-s}$ via the following substitutions:

(a) each time the connection ∇ appears in the functional $\mathcal{A}_{2k-r,2k-s}$ it is replaced by the classical instanton configuration;

(b) each time $\psi^{1,0}$ appears in $\mathcal{A}_{2k-r,2k-s}$ it is replaced by the expression

$$\psi^{1,0} = \sum_{j=1}^{n} \psi^j u_{(j)}^{1,0}(x),$$

where $\{u_{(1)}^{1,0},\ldots,u_{(n)}^{1,0}\}$ is a basis of the solution space of Eqs. (19) in the class of complex-valued one-forms of type $(1,0)$ on the chosen instanton background;

(c) each time $\psi^{0,1}$ appears in $\mathcal{A}_{2k-r,2k-s}$ it is replaced by the complex conjugate expression $\psi^{0,1} = \sum_{j=1}^{n} \overline{\psi}^j \bar{u}_{(j)}^{0,1}(x)$;

(d) each time ϕ appears in $\mathcal{A}_{2k-r,2k-s}$ it is replaced by the solution $\langle\phi\rangle$ of the equation

$$\nabla^a \nabla_a \langle\phi\rangle = i[\psi^a, \psi_a]$$

with $\psi = \psi^{1,0} + \psi^{0,1}$ replaced by its zero modes as described in items **(b)** and **(c)**.

From this prescription it follows that the map $\Phi^{r,s}: \omega_{r,s} \to \Phi_\omega^{r,s}$ translates a hypercomplex structure on M to a hypercomplex structure on the moduli space $\mathcal{M}$ and its value $\Phi_\omega^{r,s}$ on any (r,s)-form ω is a function on the moduli space $\mathcal{M}$ which is homogeneous of degree r in ψ^j and of degree s in $\overline{\psi^j}$. According to the decomposition (20), this means that $\Phi_\omega^{r,s}$ is a differential form on $\mathcal{M}$ of the same bidegree (r,s) as ω. Moreover, if $\omega = \overline{\partial}\sigma$ for some $(r,s-1)$-form σ then $\Phi_\omega^{r,s}$ is zero, since the functional

$$\int_M \mathcal{A}_{2k-r,2k-s} \wedge \overline{\partial}\sigma = \bar{q}\left((-1)^s \int_M \mathcal{A}_{2k-r,2k-s+1} \wedge \sigma \right)$$

is $\bar{q}$-exact. Thus the topological quantum field theory on a $4k$-manifold M admitting a hyperKähler structure gives the map

$$H^{r,s}(M^{4k}) \to H^{r,s}(\mathcal{M}) \qquad (33)$$
$$\omega \mapsto \Phi_{\omega}^{r,s}$$

from the Dolbeault cohomology on M to the Dolbeault cohomology on $\mathcal{M}$. In the particular case $k = 1$ this map has been constructed in Ref. [11].

Thus we conclude that despite the fact that in the intermediate considerations we used a mathematically ill-defined functional integral, we arrived finally at a very concrete and rigorous recipe for calculating hypercomplex structure invariants

$$\langle \tilde{\omega}_1 \cdots \tilde{\omega}_p \rangle = \int_{\mathcal{M}} \Phi_{\omega_1}^{r_1,s_1} \wedge \cdots \wedge \Phi_{\omega_p}^{r_p,s_p}$$

associated with any collection of non-zero Dolbeault cohomology classes, $\omega_1 \in H^{r_1,s_1}(M,\mathbb{R}), \cdots, \omega_p \in H^{r_p,s_p}(M,\mathbb{R})$, so that $r_1 + \cdots + r_p = s_1 + \cdots + s_p = \dim_{\mathbb{C}} \mathcal{M}$.

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